

# Dynamic Virtual Holonomic Constraints for Stabilization of Closed Orbits in Underactuated Mechanical Systems

Alireza Mohammadi, Manfredi Maggiore, and Luca Consolini

## Abstract

This article investigates the problem of enforcing a virtual holonomic constraint (VHC) on an underactuated mechanical system while simultaneously stabilizing a closed orbit on the constraint manifold. This problem, which to date is open, arises when designing controllers to induce complex repetitive motions in robots. In this paper, we propose a solution which relies on the parameterization of the VHC by the output of a double integrator. While the original controls are used to enforce the VHC, the control input of the double-integrator is designed to asymptotically stabilize the closed orbit and make the state of the double-integrator converge to zero. The proposed design is applied to the problem of making a PVTOL aircraft follow a circle on the vertical plane with a desired speed profile, while guaranteeing that the aircraft does not roll over.

Virtual holonomic constraints (VHCs) have been recognized to be key to solving complex motion control problems in robotics. There is an increasing body of evidence from bipedal robotics [8], [9], [26], snake robot locomotion [18], [19], [21], and repetitive motion planning [1], [5], [23] that VHCs constitute a new motion control paradigm, an alternative to the traditional reference tracking framework. The key difference with the standard motion control paradigm of robotics is that, in the VHC framework, the desired motion is parameterized by the states of the mechanical system, rather than by time.

Grizzle and collaborators (see, e.g., [26]) have shown that the enforcement of certain VHCs on a biped robot leads, under certain conditions, to the orbital stabilization of a hybrid closed orbit corresponding to a repetitive walking gait. The orbit in question lies on the constraint manifold, and the mechanism stabilizing it is the dissipation of energy that occurs when a foot impacts the ground. In a mechanical system without impacts, this stabilization mechanism disappears, and the enforcement of the VHC alone is insufficient to achieve the ultimate objective of stabilizing a repetitive behavior. Some researchers [24], [25] have addressed this problem by using the VHC exclusively for motion planning, i.e., to find a desired closed orbit. Once a suitable closed orbit is found, a time-varying controller is designed by linearizing the control system along the orbit. In this approach, the constraint manifold is not an invariant set for the closed-loop system, and thus the VHC is not enforced via feedback.

To the best of our knowledge, for mechanical control systems with degree of underactuation one, the problem of simultaneous enforcement of a VHC and orbital stabilization of a closed orbit lying on the constraint manifold is still open.

**Contributions of the paper.** This paper presents the first solution of the simultaneous stabilization problem just described. Leveraging recent results in [17], we consider VHCs that induce Lagrangian constrained dynamics. The closed orbits on the constraint manifold are level sets of a “virtual” energy function. We make the VHC dynamic by parametrizing it by the output of a double-integrator. We use the original controls of the mechanical system to enforce the dynamic VHC, while we use the double-integrator input to asymptotically stabilize the selected orbit. Because the output of the double-integrator acts as a perturbation of the original constraint manifold, we also make sure that the state of the double-integrator converges to zero. To achieve these objectives, we develop a novel theoretical result giving necessary and sufficient conditions for the exponential stabilizability of a closed orbit for a control-affine system.

The benefit of simultaneously enforcing a VHC and stabilizing a closed orbit is that it offers a superior control over the transient behavior of the system. This is illustrated in an example at the end of the paper, in which a PVTOL vehicle performs a repetitive maneuver while guaranteeing that it does not undergo full revolutions along its longitudinal axis.

**Relevant literature.** Previous work employs VHCs to stabilize desired repetitive behaviors for underactuated mechanical systems [3], [4], [24]. Canudas-de-Wit and collaborators [4] propose a technique to stabilize a desired closed orbit that relies on enforcing a virtual constraint and on dynamically changing its geometry so as to impose that the reduced dynamics on the constraint manifold match the dynamics of a nonlinear oscillator. In [3], [24], Canudas-de-Wit, Shiriaev, and collaborators employ VHCs to aid the selection of closed orbits corresponding to desired repetitive behaviors of underactuated mechanical systems. It is demonstrated that an unforced second-order system possessing an integral of motion describes the constrained motion. Assuming that this unforced system has a closed orbit, a linear time-varying controller is designed that yields exponential stability of the closed orbit. With the exception of [4], the papers above do not guarantee the invariance of the VHC for the closed loop system. The idea of event-triggered dynamic VHCs has appeared in the work by Morris and Grizzle in [20] where the authors construct a hybrid invariant manifold for the closed-loop dynamics of biped robots by updating the VHC parameters after each impact with the ground. This approach is similar in spirit to the one presented in this paper. In Section VI, we discuss the differences between the method presented in this article and the ones in [3], [4], [24]. We also discuss the conceptual similarities between the method presented in this article and the one in [20].

**Organization.** This article is organized as follows. We review preliminaries in Section I. The formal problem statement and our solution strategy are presented in Section II. In Section III we present dynamic VHCs. In Section IV we design the input of the double-integrator to stabilize the closed orbit relative to the constraint manifold, and in Section V we present the complete control law solving the VHC-based orbital stabilization problem. In Section VI we discuss the differences between the method presented in this article and the ones in [3], [4], [24]. Finally, in Section VII we apply the ideas of this paper to a path following problem for the PVTOL aircraft.

**Notation.** If  $x \in \mathbb{R}$  and  $T > 0$ , then  $x$  modulo  $T$  is denoted by  $[x]_T$ , and the set  $\{[x]_T : x \in \mathbb{R}\}$  is denoted by  $[\mathbb{R}]_T$ . This set can be given a manifold structure which makes it diffeomorphic to the unit circle  $\mathbb{S}^1$ . If  $a$  and  $b$  are vectors, then  $\text{col}(a, b) := [a^\top \ b^\top]^\top$ . If  $a, b \in \mathbb{R}^n$ , we denote  $\langle a, b \rangle = a^\top b$ , and  $\|a\| = \langle a, a \rangle^{1/2}$ .

If  $h : M \rightarrow N$  is a smooth map between smooth manifolds, and  $q \in M$ , we denote by  $dh_q : T_q M \rightarrow T_{h(q)} N$  the derivative of  $h$  at  $q$  (in coordinates, this is the Jacobian matrix of  $h$  evaluated at  $q$ ), and if  $M$  has dimension 1, then we may use the notation  $h'(q)$  in place of  $dh_q$ . If  $M_1, M_2, N$  are smooth manifolds and  $f : M_1 \times M_2 \rightarrow N$  is a smooth function, then  $\partial_{q_1} f(q_1, q_2)$  denotes the derivative of the map  $q_1 \mapsto f(q_1, q_2)$  at  $q_1$ . If  $f : M \rightarrow TM$  is a vector field on  $M$  and  $h : M \rightarrow \mathbb{R}^m$  is  $C^1$ , then  $L_f h : M \rightarrow \mathbb{R}^m$  is defined as  $L_f h(q) := dh_q f(q)$ . For a function  $h : M \rightarrow \mathbb{R}^k$ , we denote by  $h^{-1}(0) := \{q \in M : h(q) = 0\}$ .

If  $A \in \mathbb{R}^{m \times n}$  has full row-rank, we denote by  $A^\dagger$  the right-inverse of  $A$ ,  $A^\dagger = A^\top (AA^\top)^{-1}$ . Given a  $C^2$  scalar function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we denote by  $\text{Hess}(f)$  its Hessian matrix.

## I. PRELIMINARIES

Consider the underactuated mechanical control system

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla P(q) = B(q)\tau, \quad (1)$$

where  $q = (q_1, \dots, q_n) \in \mathcal{Q}$  is the configuration vector with  $q_i$  either a displacement in  $\mathbb{R}$  or an angular variable in  $[\mathbb{R}]_{T_i}$ , with  $T_i > 0$ . The configuration space  $\mathcal{Q}$  is, therefore, a generalized cylinder. In (1),  $B : \mathcal{Q} \rightarrow \mathbb{R}^{n \times n-1}$  is  $C^1$  and it has full rank  $n - 1$ . Also,  $D(q)$ , the inertia matrix, is positive definite for all  $q$ , and  $P(q)$ , the potential energy function, is  $C^1$ . We assume that there exists a left-annihilator of  $B(q)$ ; specifically, there is a  $C^1$  function  $B^\perp : \mathcal{Q} \rightarrow \mathbb{R}^{1 \times n} \setminus \{0\}$  such that  $B^\perp(q)B(q) = 0$  for all  $q \in \mathcal{Q}$ .

**Definition 1 ([15]).** A relation  $h(q) = 0$ , where  $h : \mathcal{Q} \rightarrow \mathbb{R}^k$  is  $C^2$ , is a **regular virtual holonomic constraint (VHC) of order  $k$**  for system (1), if (1) with output  $e = h(q)$  has well-defined vector relative degree  $\{2, \dots, 2\}$  everywhere on the constraint manifold

$$\Gamma := \{(q, \dot{q}) : h(q) = 0, \ dh_q \dot{q} = 0\}. \quad (2)$$

The constraint manifold  $\Gamma$  in (2) is just the zero dynamics manifold associated with the output  $e = h(q)$ . For a VHC of order  $n - 1$ , the set  $h^{-1}(0)$  is a collection of disconnected regular curves, each one diffeomorphic to either the unit circle or the real line. From now on, we will assume that  $h^{-1}(0)$  is diffeomorphic to the unit circle.

Necessary and sufficient conditions for a relation  $h(q) = 0$  to be a regular VHC of order  $n - 1$  are given in the following proposition.

**Proposition 2 ([15]).** Let  $h : \mathcal{Q} \rightarrow \mathbb{R}^{n-1}$  be  $C^2$  and such that  $\text{rank } dh_q = n - 1$  for all  $q \in h^{-1}(0)$ . Then,  $h(q) = 0$  is a regular VHC of order  $n - 1$  for system (1) if and only if for each  $q \in h^{-1}(0)$ ,

$$T_q h^{-1}(0) \oplus \text{Im}(D^{-1}(q)B(q)) = T_q \mathcal{Q}.$$

Moreover, if  $\sigma : \mathbb{R} \rightarrow \mathcal{Q}$  is a regular parameterization of  $h^{-1}(0)$ , then  $h(q) = 0$  is a regular VHC for system (1) if and only if

$$(\forall \theta \in \mathbb{R}) \ B^\perp(\sigma(\theta))D(\sigma(\theta))\sigma'(\theta) \neq 0.$$

By definition, if  $h : \mathcal{Q} \rightarrow \mathbb{R}^{n-1}$  is a regular VHC, system (1) with output  $e = h(q)$  has vector relative degree  $\{2, \dots, 2\}$ . In order to asymptotically stabilize the constraint manifold, one may employ an input-output linearizing feedback.

**Proposition 3 ([15]).** Let  $h(q) = 0$  be a regular VHC of order  $n - 1$  for system (1) with associated constraint manifold  $\Gamma$  in (2). Let  $H(q, \dot{q}) = \text{col}(h(q), dh_q \dot{q})$ , and assume that there exist two class- $\mathcal{K}$  functions  $\alpha_1, \alpha_2$  such that

$$\alpha_1(\|(q, \dot{q})\|_\Gamma) \leq H(q, \dot{q}) \leq \alpha_2(\|(q, \dot{q})\|_\Gamma). \quad (3)$$

Then, for all  $k_p, k_d > 0$ , the input-output linearizing controller

$$\tau = A^{-1}(q) \{dh_q D^{-1}(q)[C(q, \dot{q})\dot{q} + \nabla P(q)] - \mathcal{H}(q, \dot{q}) - k_p e - k_d \dot{e}\},$$

with  $e = h(q)$ ,  $\mathcal{H} = \text{col}(\mathcal{H}_1, \dots, \mathcal{H}_{n-1})$ ,  $\mathcal{H}_i = \dot{q}^\top \text{Hess}(h_i)\dot{q}$ , asymptotically stabilizes<sup>1</sup> the constraint manifold  $\Gamma$ .

<sup>1</sup> The notion of asymptotic stability of sets used in this paper is this. Consider a dynamical system  $\Sigma$  with continuous flow map  $\phi(t, x_0)$ . The set  $\Gamma$  is stable for  $\Sigma$  if for all  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $\Gamma$  such that for all  $x_0 \in U$  such that  $\phi(t, x_0)$  is defined for all  $t \geq 0$ ,  $\|\phi(t, x_0)\|_\Gamma < \varepsilon$  for all  $t \geq 0$ . The set  $\Gamma$  is asymptotically stable for  $\Sigma$  if it is stable and there exists a neighborhood  $U$  of  $\Gamma$  such that for all  $x_0 \in U$  such that  $\phi(t, x_0)$  is defined for all  $t \geq 0$ ,  $\|\phi(t, x_0)\|_\Gamma \rightarrow 0$  as  $t \rightarrow \infty$ .

Once the constraint manifold  $\Gamma$  has been rendered invariant by the above feedback, the motion on  $\Gamma$  is described by a second-order unforced differential equation, as detailed in the next proposition.

**Proposition 4** ([17]). *Let  $h(q) = 0$  be a regular VHC of order  $n - 1$  for system (1). Assume that  $h^{-1}(0)$  is diffeomorphic to the unit circle. For some  $T_1 > 0$ , let  $\sigma : [\mathbb{R}]_{T_1} \rightarrow \mathcal{Q}$  be a regular parameterization of  $h^{-1}(0)$ . Then letting  $(q, \dot{q}) = (\sigma(\theta), \sigma'(\theta)\dot{\theta})$ , the dynamics on the set  $\Gamma$  in (2) are globally described by*

$$\ddot{\theta} = \Psi_1(\theta) + \Psi_2(\theta)\dot{\theta}^2, \quad (4)$$

where  $(\theta, \dot{\theta}) \in [\mathbb{R}]_{T_1} \times \mathbb{R}$  and

$$\begin{aligned} \Psi_1(\theta) &= -\frac{B^\perp \nabla P}{B^\perp D \sigma'} \Big|_{q=\sigma(\theta)}, \\ \Psi_2(\theta) &= -\frac{B^\perp D \sigma'' + \sum_{i=1}^n B_i^\perp \sigma'^\top Q_i \sigma'}{B^\perp D \sigma'} \Big|_{q=\sigma(\theta)}, \end{aligned} \quad (5)$$

and where  $B_i^\perp$  is the  $i$ -th component of  $B^\perp$  and  $(Q_i)_{jk} = (1/2)(\partial_{q_k} D_{ij} + \partial_{q_j} D_{ik} - \partial_{q_i} D_{kj})$ .

Henceforth, we will refer to (4) as the **reduced dynamics**. System (4) is unforced since all  $n - 1$  control directions are used to make the constraint manifold  $\Gamma$  invariant.

Under certain conditions, the reduced dynamics (4) have a Lagrangian structure. Define

$$M(\theta) := \exp \left( -2 \int_0^\theta \Psi_2(\tau) d\tau \right), \quad V(\theta) := - \int_0^\theta \Psi_1(\tau) M(\tau) d\tau. \quad (6)$$

**Proposition 5** ([17]). *Consider the reduced dynamics (4) with state space  $[\mathbb{R}]_{T_1} \times \mathbb{R}$ . System (4) is Lagrangian if and only if the functions  $M(\cdot)$  and  $V(\cdot)$  in (6) are  $T_1$ -periodic, in which case the Lagrangian function is given by  $L(\theta, \dot{\theta}) = (1/2)M(\theta)\dot{\theta}^2 - V(\theta)$ .*

An immediate consequence of the foregoing result is that, when the reduced dynamics (4) are Lagrangian, the orbits of (4) are characterized by the level sets of the energy function

$$E(\theta, \dot{\theta}) = \frac{1}{2}M(\theta)\dot{\theta}^2 + V(\theta). \quad (7)$$

Moreover, almost all orbits of the reduced dynamics (4) are closed, and they belong to two distinct families, defined next.

**Definition 6.** *A closed orbit  $\gamma$  of the reduced dynamics (4) is said to be a **rotation of  $\theta$**  if  $\gamma$  is homeomorphic to a circle  $\{(\theta, \dot{\theta}) \in [\mathbb{R}]_T \times \mathbb{R} : \dot{\theta} = \text{constant}\}$  via a homeomorphism of the form  $(\theta, \dot{\theta}) \mapsto (\theta, T(\theta)\dot{\theta})$ ;  $\gamma$  is an **oscillation of  $\theta$**  if it is homeomorphic to a circle  $\{(\theta, \dot{\theta}) \in [\mathbb{R}]_{T_1} \times \mathbb{R} : \theta^2 + \dot{\theta}^2 = \text{constant}\}$  via a homeomorphism of the form  $(\theta, \dot{\theta}) \mapsto (\theta, T(\theta)\dot{\theta})$ .*

In [15, Proposition 4.7], it is shown that if the assumptions of Proposition 4 hold, then almost all orbits of (4) are either oscillations or rotations. Oscillations and rotations are illustrated in Figure I. It is possible to give an explicit regular parameterization of rotations and oscillations which will be useful in what follows.

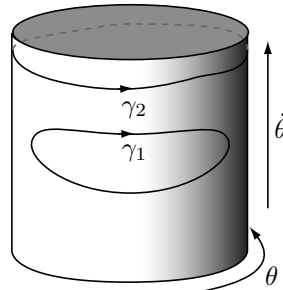


Fig. 1. An illustration of the two types of closed orbits exhibited by the reduced dynamics (4) under the assumptions of Proposition 5. The orbit  $\gamma_1$  is an oscillation, while  $\gamma_2$  is a rotation.

If  $\gamma$  is a rotation with associated energy value  $E_0$ , then we may solve  $E(\theta, \dot{\theta}) = E_0$  for  $\dot{\theta}$  obtaining

$$\dot{\theta} = \pm \sqrt{\frac{2}{M(\theta)}(E_0 - V(\theta))},$$

with plus sign for counterclockwise rotation, and minus sign for clockwise rotation. Thus a rotation  $\gamma$  is the graph of a function, which leads to the natural regular parameterization  $[\mathbb{R}]_{T_1} \rightarrow [\mathbb{R}]_{T_1 \times \mathbb{R}}$  given by

$$\vartheta \mapsto (\varphi_1(\vartheta), \varphi_2(\vartheta)) = \left( \vartheta, \pm \sqrt{\frac{2}{M(\vartheta)}(E_0 - V(\vartheta))} \right). \quad (8)$$

Concerning oscillations, it was shown in [6, Lemma 3.12] that they are mapped homeomorphically to circles via the homeomorphism

$$(\theta, \dot{\theta}) \mapsto \left( \theta, T(\theta)\dot{\theta} \right), \quad T(\theta) = \sqrt{\frac{R^2 - (\theta - C)^2}{\frac{2}{M(\theta)}(E_0 - V(\theta))}}.$$

In the above,  $E_0$  is the energy level associated with  $\gamma$ , and

$$\theta^1 := \min_{(\theta, \dot{\theta}) \in \gamma} \theta, \quad \theta^2 := \max_{(\theta, \dot{\theta}) \in \gamma} \theta, \quad C := (\theta^1 + \theta^2)/2, \quad R := (\theta^2 - \theta^1)/2.$$

The image of  $\gamma$  under the above homeomorphism is a circle of radius  $R$  centred at  $(C, 0)$ . Using this fact, we get the following regular parameterization  $[\mathbb{R}]_{2\pi} \rightarrow [\mathbb{R}]_{T_1} \times \mathbb{R}$ :

$$\vartheta \mapsto (\varphi_1(\vartheta), \varphi_2(\vartheta)) = \left( C + R \cos(\vartheta), \frac{R \sin(\vartheta)}{T(C + R \cos(\vartheta))} \right). \quad (9)$$

## II. PROBLEM STATEMENT

Consider the mechanical control system (1) with  $n$  DOFs and  $n - 1$  controls. Let  $h(q) = 0$  be a regular VHC of order  $n - 1$ , and assume that  $h^{-1}(0)$  is diffeomorphic to  $\mathbb{S}^1$ . As before, let  $\sigma : [\mathbb{R}]_{T_1} \rightarrow \mathcal{Q}$ ,  $T_1 > 0$ , be a regular parameterization of  $h^{-1}(0)$ .

Assume that the dynamics (4) are Lagrangian, so that almost all of its closed orbits are rotations or oscillations. In particular, almost every orbit of the reduced dynamics on the constraint manifold is closed, and it corresponds to a certain speed profile. Pick one such orbit of interest<sup>2</sup>,  $\gamma = \{(\theta, \dot{\theta}) : E(\theta, \dot{\theta}) = E_0\}$ . Since the reduced dynamics (4) are unforced, it is impossible to stabilize this orbit while preserving the invariance of the constraint manifold. The idea we explore in this paper is to introduce a dynamic perturbation of the constraint manifold as follows. We define a one-parameter family of VHCs  $h^s(q)$ , where  $s \in \mathbb{R}$  is the parameter and the map  $h^s(q)$  is such that  $h^0(q) = h(q)$ . Each VHC in the family can be viewed as a perturbation of the original VHC  $h(q) = 0$ . We dynamically adapt the parameter  $s$  as  $\ddot{s} = v$ , where  $v$  is a new control input to be assigned.

The constraint manifold associated with the family of VHCs  $h^s(q) = 0$  is

$$\bar{\Gamma} = \{(q, \dot{q}, s, \dot{s}) : h^s(q) = 0, \partial_q h^s \dot{q} + \partial_s h^s \dot{s} = 0\}.$$

The manifold  $\bar{\Gamma}$  can be viewed as a perturbation of the original constraint manifold  $\Gamma$ , since its intersection with the set  $\{(q, \dot{q}, s, \dot{s}) : s = \dot{s} = 0\}$  is  $\Gamma$ . The closed orbit we wish to stabilize is the curve on  $\bar{\Gamma}$  defined as

$$\bar{\gamma} := \{(q, \dot{q}, s, \dot{s}) : s = \dot{s} = 0, q = \sigma(\theta), \dot{q} = \sigma'(\theta)\dot{\theta}, (\theta, \dot{\theta}) \in [\mathbb{R}]_{T_1} \times \mathbb{R}, E(\theta, \dot{\theta}) = E_0\}.$$

**VHC-based orbital stabilization problem.** Find a smooth control law that simultaneously stabilizes the nested sets  $\bar{\gamma} \subset \bar{\Gamma}$ .

The asymptotic stabilization of  $\bar{\Gamma}$  corresponds to the enforcement of the perturbed VHC  $h^s(q) = 0$ . Since  $(s, \dot{s}) = (0, 0)$  on  $\bar{\gamma}$ , near  $\bar{\gamma}$  the Hausdorff distance between the set  $\bar{\Gamma}$  and the original constraint manifold  $\Gamma \times \{(s, \dot{s}) = (0, 0)\}$  is small. Considering the fact that  $h(q) = 0$  embodies a useful constraint that we wish to hold during the transient, the philosophy of the VHC-based orbital stabilization problem is to preserve as much as possible the beneficial properties of the original VHC  $h(q) = 0$ , while simultaneously stabilizing the closed orbit  $\gamma$  corresponding to a desired repetitive behaviour.

**Solution steps.** Our solution to the VHC-based orbital stabilization problem unfolds in three steps:

- 1) We present a technique to parameterize the VHC  $h(q) = 0$  with the output of a double integrator, giving rise to a dynamic VHC  $h^s(q) = 0$  with associated constraint manifold  $\bar{\Gamma}$ . We show that if the original VHC is regular, so too is its dynamic counterpart for small values of the double integrator output (Proposition 8). Moreover, if the original constraint manifold  $\Gamma$  is stabilizable, so too is the perturbed manifold  $\bar{\Gamma}$  (Proposition 9). We derive the reduced dynamics on this manifold, which are now affected by the input  $v$  of the double integrator.
- 2) We develop a general result for control-affine systems (Theorem 10) relating the exponential stabilizability of a closed orbit to the controllability of a linear periodic system, for which we give an explicit representation. Leveraging this result, we design the input of the double-integrator,  $v$ , to exponentially stabilize the orbit relative to the manifold  $\bar{\Gamma}$ .
- 3) We put together the controller enforcing the dynamic VHC in Step 1 with the controller stabilizing the orbit in Step 2 and show that the resulting controller solves the VHC-based orbital stabilization problem.

<sup>2</sup>Here we assume that the level set  $\{(\theta, \dot{\theta}) : E(\theta, \dot{\theta}) = E_0\}$  is connected. There is no loss of generality in this assumption, since the theory developed below relies on the regular parametrizations (8), (9), which one can use to select one of the desired connected components of  $\gamma$ .

### III. STEP 1: MAKING THE VHC DYNAMIC

In this section we present the notion of dynamic VHCs. We begin by augmenting the dynamics in (1) with a double-integrator, to obtain the augmented system

$$\begin{aligned} D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla P(q) &= B(q)\tau, \\ \ddot{s} &= v. \end{aligned} \quad (10)$$

Henceforth, we use overbars to distinguish objects associated with the augmented control system (10) from those associated with (1). Accordingly, we define  $\bar{q} := (q, s)$ ,  $\dot{\bar{q}} := (\dot{q}, \dot{s})$ ,  $\bar{\mathcal{Q}} := \{(q, s) : q \in \mathcal{Q}, s \in \mathcal{I}\}$ .

**Definition 7.** Let  $h(q) = 0$  be a regular VHC of order  $n - 1$  for system (1). A **dynamic VHC based on**  $h(q) = 0$  is a relation  $h^s(q) = 0$  such that the map  $(s, q) \mapsto h^s(q)$  is  $C^2$ ,  $h^0(q) = h(q)$ , and the parameter  $s$  satisfies the differential equation  $\ddot{s} = v$  in (10).

The dynamic VHC  $h^s(q)$  is **regular** for (10) if there exists an open interval  $\mathcal{I} \subset \mathbb{R}$  containing  $s = 0$  such that, for all  $s \in \mathcal{I}$  and all  $v \in \mathbb{R}$ , system (10) with input  $\tau$  and output  $e = h^s(q)$  has vector relative degree  $\{2, \dots, 2\}$ .

The dynamic VHC  $h^s(q) = 0$  is **stabilizable** for (10) if there exists a smooth feedback  $\tau(q, \dot{q}, s, \dot{s}, v)$  such that the manifold

$$\bar{\Gamma} := \{(q, \dot{q}, s, \dot{s}) : h^s(q) = 0, \partial_q h^s \dot{q} + \partial_s h^s \dot{s} = 0\}, \quad (11)$$

is asymptotically stable for the closed-loop system.

The reason for parameterizing the VHC with the output of a double integrator is to guarantee that the input  $v$  of the double integrator appears after taking two derivatives of the output function  $e = h^s(q)$ . The regularity property of  $h^s(q)$  in the foregoing definition means that, upon calculating the second derivative of  $e = h^s(q)$  along the vector field in (10), the control input  $\tau$  appears nonsingularly, i.e.,

$$\ddot{e} = (\star) + A^s(q)\tau + B^s(q)v,$$

where  $A^s$  and  $B^s$  are suitable matrices, and  $A^s$  is invertible for all  $q \in (h^s)^{-1}(0)$  and all  $s \in \mathcal{I}$ .

Given a regular VHC  $h(q) = 0$ , a possible way to generate a dynamic VHC based on  $h(q) = 0$  is to translate the curve  $h^{-1}(0)$  by an amount proportional to  $s \in \mathbb{R}$ . Other choices are of course possible, but this one has the benefit of allowing for simple expressions in the derivations that follow. We thus consider the following one-parameter family of mappings

$$h^s(q) := h(q - Ls), \quad (12)$$

where  $L \in \mathbb{R}^n$  is a non-zero constant vector. The zero level set of each family member in (12) is  $(h^s)^{-1}(0) = \{q + Ls : q \in h^{-1}(0)\}$ , a translation<sup>3</sup> of  $h^{-1}(0)$  by the vector  $Ls$  (see Figure 2). If  $\sigma : [\mathbb{R}]_{T_1} \rightarrow \mathcal{Q}$  is a regular parameterization of the curve  $h^{-1}(0)$ , a regular parameterization of the zero level set of each family member in (12) is  $\sigma^s(\theta) = \sigma(\theta) + Ls$ . In an analogous manner, the constraint manifold  $\bar{\Gamma}$  in (11) is the translation of  $\Gamma$  by the vector  $\text{col}(Ls, L\dot{s})$ ,

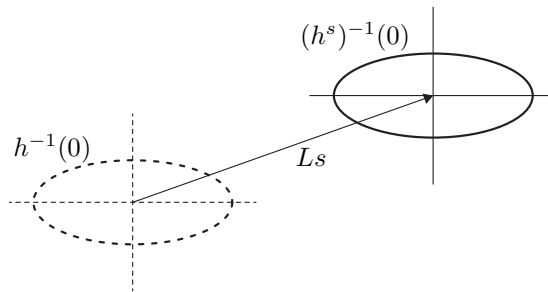


Fig. 2. Geometric interpretation of the dynamic VHC (12).

$$\begin{aligned} \bar{\Gamma} &= \{(q, \dot{q}, s, \dot{s}) : h(q - Ls) = 0, dh_{q-Ls}(\dot{q} - L\dot{s}) = 0\} \\ &= \{(q, \dot{q}, s, \dot{s}) : (q - Ls, \dot{q} - L\dot{s}) \in \Gamma\}. \end{aligned} \quad (13)$$

In the augmented coordinates, the closed orbit we wish to stabilize is

$$\bar{\gamma} = \{(q, \dot{q}, s, \dot{s}) : s = \dot{s} = 0, q = \sigma(\theta), \dot{q} = \sigma'(\theta)\dot{\theta}, (\theta, \dot{\theta}) \in [\mathbb{R}]_{T_1} \times \mathbb{R}, E(\theta, \dot{\theta}) = E_0\}. \quad (14)$$

The next two propositions show that if  $h(q) = 0$  is regular and stabilizable, so too is its dynamic counterpart  $h(q - Ls) = 0$ .

<sup>3</sup>Recall that  $q$  is a  $n$ -tuple whose  $i$ -th element,  $q_i$ , is either a real number or an element of  $[\mathbb{R}]_{T_i}$ . In the latter case, the sum  $q_i + L_i s$  is to be understood as sum modulo  $T_i$ .

**Proposition 8.** If  $h(q) = 0$  is a regular VHC of order  $n - 1$  for (1), then for any  $L \in \mathbb{R}^n$  the dynamic VHC  $h(q - Ls) = 0$  is regular for the augmented system (10).

**Proposition 9.** If  $h(q) = 0$  is a regular VHC of order  $n - 1$  for (1) satisfying the stabilizability condition (3), then for any  $L \in \mathbb{R}^n$ , the dynamic VHC  $h(q - Ls) = 0$  is stabilizable in the sense of Definition 7, and a feedback stabilizing the constraint manifold  $\bar{\Gamma}$  in (13) is  $\tau = \tau^*(q, \dot{q}, s, \dot{s}, v)$  with

$$\tau^*(q, \dot{q}, s, \dot{s}, v) = (A^s(q))^{-1} \left\{ dh_{q-Ls} D^{-1}(q) [C(q, \dot{q})\dot{q} + \nabla P(q)] + dh_{q-Ls} Lv - \mathcal{H}(q, \dot{q}, s, \dot{s}) - k_p e - k_d \dot{e} \right\}, \quad (15)$$

where  $e = h(q - Ls)$ ,  $\dot{e} = dh_{q-Ls}(\dot{q} - L\dot{s})$ ,  $A^s(q) = dh_{q-Ls} D^{-1}(q)B(q)$ ,  $k_p, k_d > 0$ , and  $\mathcal{H} = \text{col}(\mathcal{H}_1, \dots, \mathcal{H}_{n-1})$ ,  $\mathcal{H}_i = (\dot{q} - L\dot{s})^\top \text{Hess}(h_i)|_{q-Ls}(\dot{q} - L\dot{s})$ .

Next, we find the reduced dynamics of the augmented system (10) with feedback (15) on the manifold  $\bar{\Gamma}$  in (13). To this end, we left-multiply (10) by the left annihilator  $B^\perp$  of  $B$  and evaluate the resulting equation on  $\bar{\Gamma}$  by setting

$$q = \sigma(\theta) + Ls, \quad \dot{q} = \sigma'(\theta)\dot{\theta} + L\dot{s}, \quad \ddot{q} = \sigma''(\theta)\ddot{\theta} + \sigma''(\theta)\dot{\theta}^2 + Lv.$$

By so doing, one obtains:

$$\begin{aligned} \ddot{\theta} &= \Psi_1^s(\theta) + \Psi_2^s(\theta)\dot{\theta}^2 + \Psi_3^s(\theta)\dot{\theta}\dot{s} + \Psi_4^s(\theta)\dot{s}^2 + \Psi_5^s(\theta)v, \\ \ddot{s} &= v, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Psi_1^s(\theta) &= - \frac{B^\perp \nabla P}{B^\perp D \sigma'} \Big|_{q=\sigma(\theta)+Ls}, \\ \Psi_2^s(\theta) &= - \frac{B^\perp D \sigma'' + \sum_{i=1}^n B_i^\perp \sigma'^\top Q_i \sigma'}{B^\perp D \sigma'} \Big|_{q=\sigma(\theta)+Ls}, \\ \Psi_3^s(\theta) &= - \frac{2 \sum_{i=1}^n B_i^\perp \sigma'^\top Q_i L}{B^\perp D \sigma'} \Big|_{q=\sigma(\theta)+Ls}, \\ \Psi_4^s(\theta) &= - \frac{\sum_{i=1}^n B_i^\perp L^\top Q_i L}{B^\perp D \sigma'} \Big|_{q=\sigma(\theta)+Ls}, \\ \Psi_5^s(\theta) &= - \frac{B^\perp D L}{B^\perp D \sigma'} \Big|_{q=\sigma(\theta)+Ls}. \end{aligned} \quad (17)$$

The fourth-order differential equation (16) will be henceforth referred to as the **extended reduced dynamics** induced by the dynamic VHC  $h(q - Ls) = 0$ . It represents the motion of the mechanical system (1) on the constraint manifold  $\bar{\Gamma}$  in (13). Its restriction to the plane  $\{s = \dot{s} = 0\}$  coincides with the reduced dynamics (4). The state  $(\theta, \dot{\theta}, s, \dot{s}) \in [\mathbb{R}]_{T_1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  represents global coordinates for  $\bar{\Gamma}$ . Using these coordinates, and with a slight abuse of notation, the closed orbit  $\bar{\gamma}$  in (14) is given by

$$\bar{\gamma} = \{(\theta, \dot{\theta}, s, \dot{s}) \in [\mathbb{R}]_{T_1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : E(\theta, \dot{\theta}) = E_0, s = \dot{s} = 0\}. \quad (18)$$

This is the set we will stabilize next.

#### IV. STEP 2: LINEARIZATION ALONG THE CLOSED ORBIT

The objective now is to design the control input  $v$  in the extended reduced dynamics (16) so as to stabilize the closed orbit  $\bar{\gamma}$  in (18). We will do so by adopting the philosophy of Hauser et al. in [12] that relies on an implicit representation of the closed orbit to derive the so-called transverse linearization along  $\bar{\gamma}$ . Roughly speaking, this is the linearization along  $\bar{\gamma}$  of the components of the dynamics that are transversal to  $\bar{\gamma}$ . Hauser's approach generalizes classical results of Hale [11, Chapter VI], who require a moving orthonormal frame. The insight in [12] is that orthogonality is not needed, transversality is enough. This insight allowed Hauser et al. in [12] to derive a normal form analogous to that in [11, Chapter VI], but calculated directly from an implicit representation of the orbit. We shall use the same idea in the theorem below.

We begin by enhancing the results of [11], [12] in two directions. First, while [11], [12] require the knowledge of a periodic solution, we only require a parameterization of  $\bar{\gamma}$  (something that is readily available in the setting of this paper, while the solution is not). Second, while [11], [12] deals with dynamics without inputs, we provide a necessary and sufficient criterion for the exponential stabilizability of the orbit.

**A general result.** Our first result is a necessary and sufficient condition for a closed orbit to be exponentially stabilizable. This result is of considerable practical use, and is of independent interest.

Consider a control-affine system

$$\dot{x} = f(x) + g(x)u, \quad (19)$$

with state  $x \in \mathcal{X}$ , where  $\mathcal{X}$  is a closed embedded submanifold of  $\mathbb{R}^n$ , and control input  $u \in \mathbb{R}^m$ . A closed orbit  $\gamma$  is **exponentially stabilizable** for (19) if there exists a locally Lipschitz continuous feedback  $u^*(x)$  such that the set  $\gamma$  is exponentially stable for the closed-loop system  $\dot{x} = f(x) + g(x)u^*(x)$ , i.e., there exist  $\delta, \lambda, M > 0$  such that for all  $x_0 \in \mathcal{X}$  such that  $\|x_0\|_\gamma < \delta$ , the solution  $x(t)$  of the closed-loop system satisfies  $\|x(t)\|_\gamma \leq M\|x_0\|_\gamma e^{-\lambda t}$  for all  $t \geq 0$ . Note that if  $\gamma$  is exponentially stable, then  $\gamma$  is asymptotically stable.

Let  $T$  be a positive real number. A linear  $T$ -periodic system  $dx/dt = A(t)x$ , where  $A(\cdot)$  is a continuous and  $T$ -periodic matrix-valued function, is **asymptotically stable** if all its characteristic multipliers lie in the open unit disk. A linear  $T$ -periodic control system

$$\frac{dx}{dt} = A(t)x + B(t)u, \quad (20)$$

where  $A(\cdot)$  and  $B(\cdot)$  are continuous and  $T$ -periodic matrix-valued functions, is **stabilizable** (or the pair  $(A(\cdot), B(\cdot))$  is stabilizable) if there exists a continuous and  $T$ -periodic matrix-valued function  $K(\cdot)$  such that  $\dot{x} = (A(t) + B(t)K(t))x$  is asymptotically stable. In this case, we say that the feedback  $u = K(t)x$  **stabilizes** system (20). The notion of stabilizability can be characterized in terms of the characteristic multipliers of  $A(\cdot)$  (see, e.g. [2]).

**Theorem 10.** Consider system (19), where  $f$  is a  $C^1$  vector field and  $g$  is locally Lipschitz continuous on  $\mathcal{X}$ . Let  $\gamma \subset \mathcal{X}$  be a closed orbit of the open-loop system  $\dot{x} = f(x)$ , and let  $\vartheta \mapsto \varphi(\vartheta)$ ,  $[\mathbb{R}]_T \rightarrow \mathcal{X}$ , be a regular parameterization of  $\gamma$ . Finally, let  $H : \mathcal{X} \rightarrow \mathbb{R}^{n-1}$  be an implicit representation of  $\gamma$  with the properties that  $H$  is  $C^1$ ,  $\text{rank } dH_x = n - 1$  for all  $x \in H^{-1}(0)$ , and  $H^{-1}(0) = \gamma$ .

(a) The orbit  $\gamma$  is exponentially stabilizable for (19) if, and only if, the linear  $T$ -periodic control system on  $\mathbb{R}^{n-1}$

$$\begin{aligned} \dot{z} &= A(t)z + B(t)u \\ A(t) &= \frac{\|\varphi'(t)\|^2}{\langle f(\varphi(t)), \varphi'(t) \rangle} \left[ (dL_f H)_{\varphi(t)} dH_{\varphi(t)}^\dagger \right] \\ B(t) &= \frac{\|\varphi'(t)\|^2}{\langle f(\varphi(t)), \varphi'(t) \rangle} \left[ L_g H(\varphi(t)) \right], \end{aligned} \quad (21)$$

is stabilizable.

(b) If a  $T$ -periodic feedback  $u = K(t)z$ , with  $K(\cdot)$  continuous and  $T$ -periodic, stabilizes the  $T$ -periodic system (21), then for any smooth map  $\pi : \mathcal{U} \rightarrow [\mathbb{R}]_T$ , with  $\mathcal{U}$  a neighborhood of  $\gamma$  in  $\mathcal{X}$  and  $\pi|_\Gamma = \varphi^{-1}$ , the feedback

$$u^*(x) = K(\pi(x))H(x) \quad (22)$$

exponentially stabilizes the closed orbit  $\gamma$  for (19).

The proof of Theorem 10 is found in the appendix.

**Remark 11.** Concerning the existence of the function  $H$  in the theorem statement, since closed orbits of smooth dynamical systems are diffeomorphic to the unit circle  $\mathbb{S}^1$ , it is always possible to find a function  $H$  satisfying the assumptions of the theorem. This well-known fact is shown, e.g., in [12, Proposition 1.2]. As for the existence of the function  $\pi : \mathcal{U} \rightarrow [\mathbb{R}]_T$ , this function can be constructed by picking  $\mathcal{U}$  to be a tubular neighborhood of  $\gamma$ . Then there exists a smooth retraction  $r : \mathcal{U} \rightarrow \gamma$ . The function  $\pi = \varphi^{-1} \circ r$  has the desired properties.  $\triangle$

**Remark 12.** Theorem 10 establishes the equivalence between the exponential stabilizability of the closed orbit  $\gamma$  and the stabilizability of the linear periodic system (21), the so-called transverse linearization. The equivalence between these two concepts is not new, it is essentially contained in the results of [11, Chapter VI] and [12]. What is new in Theorem 10, and of considerable practical interest, is the fact that it provides an explicit expression for the transverse linearization that can be computed using any regular parametrization  $\varphi$  of  $\gamma$  and any implicit representation  $H$  of  $\gamma$  whose Jacobian matrix has full rank on  $\gamma$ . In contrast to the above, the methods in [11] and [12] rely on the knowledge of a periodic open-loop solution of (19) generating  $\gamma$  and do not give an explicit expression for  $(A(\cdot), B(\cdot))$ . We also mention that Hauser's notion of transverse linearization was applied in [24] to a special class of systems, once again requiring the knowledge of a periodic solution.

**Design of the  $T$ -periodic feedback matrix  $K$ .** Once it is established that the pair  $(A(\cdot), B(\cdot))$  in (21) is stabilizable, the design of the  $T$ -periodic feedback matrix  $K(\cdot)$  in part (b) of the theorem can be carried out by solving the periodic Riccati equation

$$-\frac{d\Pi}{dt} = A(t)^\top \Pi(t) + \Pi(t)A(t) - P(t)B(t)R^{-1}B(t)^\top \Pi(t) + Q(t). \quad (23)$$

where  $R(\cdot) = R(\cdot)^\top$  is a positive definite continuous matrix-valued function and  $Q(\cdot) = Q(\cdot)^\top$  is a positive definite continuous matrix-valued function, and setting

$$K(t) = -\frac{1}{R}B(t)^\top \Pi(t). \quad (24)$$

Theorem 6.5 in [2] states that if, and only if,  $(A(\cdot), B(\cdot))$  is stabilizable and  $(Q^{1/2}(\cdot), A(\cdot))$  is detectable (this latter condition is satisfied, e.g., by letting  $Q$  be the identity matrix) then the Riccati equation (23) has a unique positive semidefinite  $T$ -periodic solution  $\Pi(\cdot)$  and the feedback  $u = K(t)z$ , with  $K(\cdot)$  given in (24), stabilizes system (21). Once this is done, the feedback  $u^*(x)$  in (22) exponentially stabilizes the closed orbit  $\gamma$ .

**Application to extended reduced dynamics.** We now apply Theorem 10 to the extended reduced dynamics (16) with the objective of stabilizing the closed orbit  $\bar{\gamma}$  in (18). Here we have  $x = (\theta, \dot{\theta}, s, \dot{s})$ , and the implicit representation of  $\bar{\gamma}$

$$H(x) = (E(\theta, \dot{\theta}) - E_0, s, \dot{s}).$$

Leveraging the parameterizations of closed orbits presented in Section I, the parameterization of  $\bar{\gamma}$  in  $(\theta, \dot{\theta}, s, \dot{s})$ -coordinates has the form  $[\mathbb{R}]_{T_2} \rightarrow [\mathbb{R}]_{T_1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ ,  $\vartheta \mapsto (\varphi_1(\vartheta), \varphi_2(\vartheta), 0, 0)$ , with  $\varphi_1, \varphi_2$  given by (8) and  $T_2 = T_1$  if  $\gamma$  is a rotation; and  $\varphi_2$  given by (9) and  $T_2 = 2\pi$  if  $\gamma$  is an oscillation. Applying Theorem 10 to system (16), we get the following  $T_2$ -periodic linear system

$$\dot{z} = \begin{bmatrix} 0 & a_{12}(t) & a_{13}(t) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} b_1(t) \\ 0 \\ 1 \end{bmatrix} v, \quad (25)$$

where

$$\begin{aligned} a_{12}(t) &= \eta(t)M(\varphi_1(t))\varphi_2(t) \left[ \partial_{z_2} \Psi_1^{z_2}(\varphi_1(t)) + \partial_{z_2} \Psi_2^{z_2}(\varphi_1(t))\varphi_2^2(t) \right] \Big|_{z_2=0}, \\ a_{13}(t) &= \eta(t)M(\varphi_1(t))\varphi_2^2(t)\Psi_3^0(\varphi_1(t)), \\ b_1(t) &= \eta(t)M(\varphi_1(t))\varphi_2(t)\Psi_5^0(\varphi_1(t)), \\ \eta(t) &= \frac{\varphi_1^2(t) + \varphi_2^2(t)}{\varphi_1'(t)\varphi_2(t) + \varphi_2'(t)[\Psi_1(\varphi_1(t)) + \Psi_2(\varphi_1(t))\varphi_2^2(t)]}. \end{aligned} \quad (26)$$

Assuming that system (25) is stabilizable, then we may find the unique positive semidefinite solution of the periodic Riccati equation (23) to get the matrix-valued function  $K(\cdot)$  in (24). Theorem 10 guarantees that the controller

$$v = \bar{v}(\theta, \dot{\theta}, s, \dot{s}) = K(\pi(\theta, \dot{\theta}, s, \dot{s})) \begin{bmatrix} E(\theta, \dot{\theta}) - E_0 \\ s \\ \dot{s} \end{bmatrix} \quad (27)$$

exponentially stabilizes the orbit  $\bar{\gamma}$  in (18) for the extended reduced dynamics (16).

It remains to find an explicit expression for the map  $\pi$ . If  $\gamma$  is a rotation, then in light of the parameterization (8), we may set

$$\pi(\theta, \dot{\theta}, s, \dot{s}) = \theta.$$

Else, if  $\gamma$  is an oscillation, using (9) we set

$$\pi(\theta, \dot{\theta}, s, \dot{s}) = \text{atan2}(T(\theta)\dot{\theta}, \theta - C),$$

where  $\text{atan2}(\cdot, \cdot)$  is the four-quadrant arctangent function such that  $\text{atan2}(\sin(\alpha), \cos(\alpha)) = \alpha$  for all  $\alpha \in (-\pi, \pi)$ .

## V. STEP 3: SOLUTION OF THE VHC-BASED ORBITAL STABILIZATION PROBLEM

In Section III, we designed the feedback  $\tau^*$  in (15) to asymptotically stabilize the constraint manifold  $\bar{\Gamma}$  associated with the dynamic VHC  $h(q - Ls) = 0$ . In Section IV, we designed the feedback  $\bar{v}$  in (27) for the double integrator  $\ddot{s} = v$  rendering the closed orbit  $\bar{\gamma}$  exponentially stable relative to  $\bar{\Gamma}$  (i.e., when initial conditions are on  $\bar{\Gamma}$ ). There are two things left to do in order to solve the VHC-based orbital stabilization problem. First, in order to implement the feedback  $\bar{v}$  in (27), we need to relate the variables  $(\theta, \dot{\theta})$  to the state  $(q, \dot{q})$ . Second, we need to show that the asymptotic stability of  $\bar{\Gamma}$  and the asymptotic stability of  $\bar{\gamma}$  relative to  $\bar{\Gamma}$  imply that  $\bar{\gamma}$  is asymptotically stable.

To address the first issue, we leverage the fact that, since  $h^{-1}(0)$  is a closed embedded submanifold of  $\mathcal{Q}$ , by [14, Proposition 6.25] there exists a neighborhood  $\mathcal{W}$  of  $h^{-1}(0)$  in  $\mathcal{Q}$  and a smooth retraction of  $\mathcal{W}$  onto  $h^{-1}(0)$ , i.e., a smooth map  $r : \mathcal{W} \rightarrow h^{-1}(0)$  such that  $r|_{h^{-1}(0)}$  is the identity on  $h^{-1}(0)$ . Define  $\Theta : \mathcal{W} \rightarrow [\mathbb{R}]_T$  as  $\Theta = \sigma^{-1} \circ r$ . By construction,  $\Theta|_{h^{-1}(0)} = \sigma^{-1}$ . In other words, for all  $q \in h^{-1}(0)$ ,  $\Theta(q)$  gives that unique value of  $\theta \in [\mathbb{R}]_T$  such that  $q = \sigma(\theta)$ . Using the function  $\Theta$ , we now define an extension of  $\bar{v}$  from  $\bar{\Gamma}$  to a neighborhood of  $\bar{\Gamma}$  as follows

$$v^*(q, \dot{q}, s, \dot{s}) = \bar{v}(\theta, \dot{\theta}, s, \dot{s}) \Big|_{(\theta, \dot{\theta}) = (\Theta(q), d\Theta_q \dot{q})}. \quad (28)$$

We are now ready to solve the VHC-based orbital stabilization problem.



**Theorem 13.** Consider system (1) and let  $h(q) = 0$  be a regular VHC of order  $n - 1$ . Let  $\sigma : [\mathbb{R}]_{T_1} \rightarrow \mathcal{Q}$  be a regular parametrization of  $h^{-1}(0)$  and consider the following assumptions:

- (a) The VHC  $h(q) = 0$  satisfies the stabilizability condition (3).
- (b) The VHC  $h(q) = 0$  induces Lagrangian reduced dynamics as per Proposition 5.
- (c) For a closed orbit  $\gamma$  of the reduced dynamics given in implicit form as  $\gamma = \{(\theta, \dot{\theta}) \in [\mathbb{R}]_{T_1} \times \mathbb{R} : E(\theta, \dot{\theta}) = E_0\}$ , consider one of the regular parametrizations  $[\mathbb{R}]_{T_2} \mapsto [\mathbb{R}]_{T_1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  discussed in Section IV. Assume that the  $T_2$ -periodic system (25)-(26) is stabilizable.

Under the assumptions above, let  $Q(\cdot) = Q(\cdot)^\top$  be a positive definite  $T_2$ -periodic  $\mathbb{R}^{3 \times 3}$ -valued function such that  $(Q^{1/2}, A)$  is detectable, and pick any  $R > 0$ . The smooth dynamic feedback

$$\begin{aligned}\tau &= \tau^*(q, \dot{q}, s, \dot{s}, v^*(q, \dot{q}, s, \dot{s})) \\ \ddot{s} &= v^*(q, \dot{q}, s, \dot{s}),\end{aligned}$$

with  $\tau^*$  defined in (15),  $v^*$  defined in (27), (28), and where  $K(\cdot)$  in (24) results from the solution of the  $T_2$ -periodic Riccati equation (23), stabilizes the nested sets  $\bar{\gamma} \subset \bar{\Gamma}$  given in (13), (14).

*Proof.* Propositions 8 and 9 establish that the feedback (15) stabilizes the set  $\bar{\Gamma}$ . Theorem 6.5 in [2] establishes that  $K(\cdot)$  in (24) is well-defined, and Theorem 10 establishes that  $v^*$  in (28) stabilizes  $\bar{\gamma}$  relative to  $\bar{\Gamma}$ . Since  $\bar{\gamma}$  is a compact set, the reduction theorem for stability of compact sets in [22], [7] implies that  $\bar{\gamma}$  is asymptotically stable for the closed-loop system.  $\square$

The block diagram of the VHC-based orbital stabilizer is depicted in Figure 3.

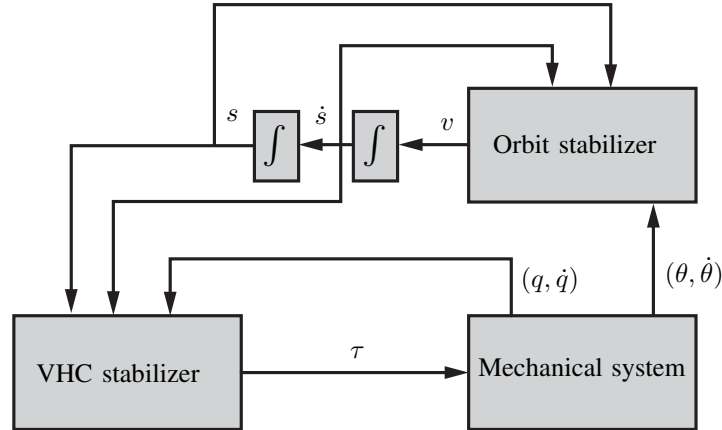


Fig. 3. Block diagram of the VHC-based orbital stabilizer.

## VI. DISCUSSION

In this section we briefly compare the control methodology of this paper with the ones in [3], [4], [20], [24].

**Comparison with [20].** The notion of dynamic hybrid extension introduced by Morris and Grizzle in [20] bears a conceptual resemblance to dynamic VHCs and their extended reduced dynamics presented in this article. In [20], the VHCs that induce stable walking gaits of biped robots are parameterized using variables whose evolution are event-triggered. In particular, the VHC parameters get updated after each impact of the swing leg with the ground. The update law is designed such that the invariance of a suitably modified manifold, which the authors call the extended zero dynamics manifold, is preserved while simultaneously enforcing a periodic stable walking gait on the biped. Our approach follows the same philosophy of preserving the invariance of a suitably modified manifold in order to maintain the desired configurations of the mechanical system. However, in our framework, the dynamics of the VHC parameter are continuous rather than event-triggered.

**Comparison with [4].** The approach by Canudas-de-wit et al. in [4] also relies on dynamically changing the geometry of VHCs. A target orbit on the constraint manifold, which is generated by a harmonic oscillator, is considered and the dynamics of the VHC parameter is designed such that the target orbit is stabilized on the constraint manifold. This approach, however, cannot be used to stabilize an assigned closed orbit induced by the original VHC on the constraint manifold. Moreover, the methodology in [4] has only been employed to control the periodic motions of a pendubot. It is unclear to what extent it can be generalized to other mechanical systems.

**Comparison with [3], [24].** In [3], [24], the authors employ VHCs to find feasible closed orbits of underactuated mechanical systems. Once the orbit is found, it is stabilized through transverse linearization of the  $2n$ -dimensional dynamics (1) along

the closed orbit. Similarly to this paper, in [3], [24] the stabilization of the transverse linearization is carried out by solving a periodic Riccati equation. But while the linearized system in [3], [24] has dimension  $2n - 1$ , the linearized system (25) always has dimension 3. And while the feedback in Theorem 13 is time-independent, the one proposed in [3], [24] is time-varying. Additionally, while the approach proposed in this paper gives explicit parametrizations of the orbits to be stabilized, the approaches in [3], [24] require the knowledge of the actual periodic trajectory which is not available in analytic form. The most important difference between the approach in this paper and the ones in [3], [24] lies in the fact that, in [3], [24], the time-varying controller does not preserve the invariance of the constraint manifold.

## VII. EXAMPLE

In this section we use the theory developed in this paper to enhance a result found in [6]. We consider the model of a V/STOL aircraft in planar vertical take-off and landing mode (PVTOL), introduced by Hauser et al. in [13]. The vehicle in question is depicted in Figure 4, where it is assumed that a preliminary feedback has been designed making the centre of mass of the aircraft lie on a unit circle on the vertical plane,  $\mathcal{C} = \{x \in \mathbb{R}^2 : |x| = 1\}$ , also depicted in the figure. In [6] it was shown that the model of the aircraft on the circle is given by

$$\begin{aligned}\ddot{q}_1 &= \frac{\mu}{\epsilon} (g \sin(q_1) - \cos(q_1 - q_2) \dot{q}_2^2 + \sin(q_1 - q_2) u), \\ \ddot{q}_2 &= u,\end{aligned}\tag{29}$$

where  $q_1$  denotes the roll angle,  $q_2$  the angular position of the aircraft on the circle, and  $u$  the so-called tangential control input resulting from the design in [6]. Also,  $\mu$  and  $\epsilon$  are positive constants. In this example, we set  $\mu/\epsilon = 1$ .

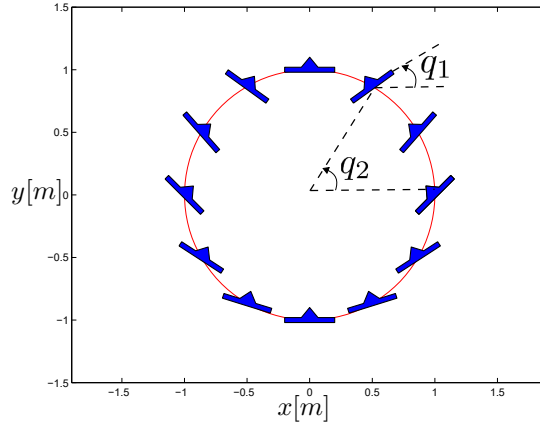


Fig. 4. Configurations of a PVTOL vehicle on the unit circle under the VHC proposed in [6].

In [6], a feedback  $u(q, \dot{q})$  was designed to enforce a regular VHC of the form  $h(q) = q_1 - f(q_2) = 0$ , represented in Figure 4. It was shown that the ensuing reduced dynamics, a few orbits of which are depicted in Figure 5, are Lagrangian. Each closed orbit in Figure 5 represents a motion of the PVTOL on the circle, with roll angle  $q_1$  constrained to be a function of the position,  $q_2$ , on the circle. Orbits in the shaded area represent a rocking motion of the PVTOL along the circle (these are oscillations), while orbits in the unshaded area represent full traversal of the circle (these are rotations). The theory in [6] was unable to stabilize individual closed orbits of the reduced dynamics. The theory of this paper fills the gap left in [6].

We wish to stabilize the closed orbit  $\gamma^+$  depicted in Figure 5 which corresponds to the energy level set  $E_0 = 41.5$ . We render the VHC dynamic by setting  $h^s(q) = q_1 - L_1 s - f(q_2 - L_2 s) = 0$ , with  $L = \text{col}(L_1, L_2) = \text{col}(0.1, 0.1)$ . Since  $\gamma^+$  is a rotation, we parameterize with the map (8). We check numerically that the pair  $(A(t), B(t))$  in (25), (26) is controllable, and pick  $R = 1$  and  $Q = \text{diag}\{1, 1, 1\}$  to set up the Riccati equation (23). We numerically solve this equation and find the gain matrix  $K(\cdot)$ . The simulation results for the controller in Theorem 13 are presented next.

Figures 6 and 7 depict the graph of the function  $h^{s(t)}(q(t))$  and the output of the double integrator,  $s(t)$ , respectively. They reveal that the VHC is properly enforced and that  $s(t) \rightarrow 0$ . Figures 8 and 9 depict the energy of the vehicle on the constraint manifold and the time trajectory of  $(\theta(t), \dot{\theta}(t))$  on the cylinder  $\mathbb{S}^1 \times \mathbb{R}$ . The energy level  $E_0$  is stabilized and the trajectory

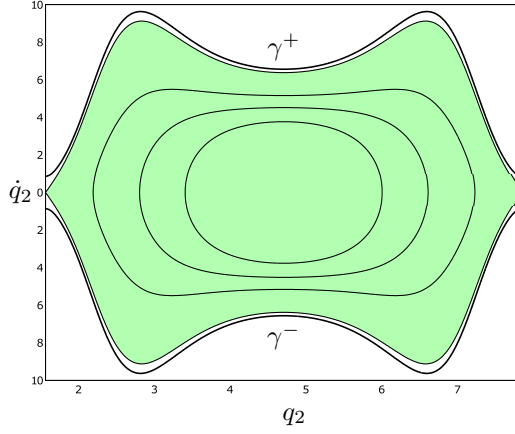


Fig. 5. The phase portrait of the reduced dynamics of the PVTOL vehicle under the VHC depicted in Figure 4. The closed orbits in the shaded area correspond to oscillations. The rest of the orbits correspond to rotations. We would like to stabilize the counterclockwise rotation  $\gamma^+$  corresponding to the energy level set  $E_0 = 41.5$ .

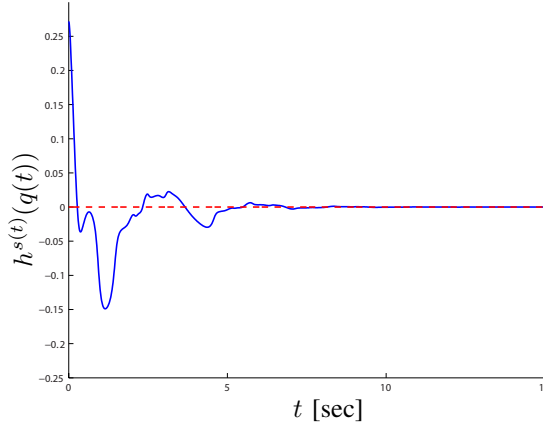


Fig. 6. The VHC  $h(q) = 0$  is asymptotically stabilized on the vehicle.

on the cylinder converges to  $\gamma^+$ . Finally, Figure 10 depicts the graph of the roll angle  $q_1(t)$ , demonstrating that, due to the enforcement of the dynamic version of the VHC depicted in Figure 4, the vehicle does not turn over.

### VIII. CONCLUSIONS

We have proposed a technique to enforce a VHC on a mechanical control system and simultaneously stabilize a closed orbit on the constraint manifold. The theory of this paper is applicable to mechanical control systems with degree of underactuation one. For higher degrees of underactuation, the reduced dynamics are described by a differential equation of order higher than two and, generally, the problem of characterizing closed orbits becomes harder. The result of Section IV concerning the exponential stabilization of closed orbits for control-affine systems is still applicable in this case.

### APPENDIX

**Proof of Proposition 8.** Considering the output  $e = h(q - Ls)$  and taking two derivatives along system (10), we get

$$\ddot{e} = (\star) - dh|_{q-Ls}Lv + A^s(q)\tau,$$

where  $A^s(q) = dh_{q-Ls}D^{-1}(q)B(q)$ . Denote  $\mu^s := \min_{q \in (h^s)^{-1}(0)} \det A^s(q)$ . Then,  $s \mapsto \mu^s$  is a continuous function. We claim that  $\mu^0 \neq 0$ . Indeed, the assumption that  $h(q) = 0$  is regular implies by Proposition 2 that  $\det A^0(q) \neq 0$  for all  $q \in (h^0)^{-1}(0) = h^{-1}(0)$ . Since  $h^{-1}(0)$  is a compact set and  $q \mapsto \det A^0(q)$  is continuous,  $\min(\det A^0(q)) \neq 0$ , proving that  $\mu^0 \neq 0$ , as claimed. By continuity, there exists an open interval  $\mathcal{I} \subset \mathbb{R}$  containing  $s = 0$  such that  $\mu^s \neq 0$  on  $\mathcal{I}$  implying that  $A^s(q)$  is nonsingular for all  $q \in (h^s)^{-1}(0)$  and all  $s \in \mathcal{I}$ .  $\square$

**Proof of Proposition 9.** By (13), we have  $\bar{\Gamma} = \{(q, \dot{q}, s, \dot{s}) \in T\bar{\mathcal{Q}} : (q - Ls, \dot{q} - L\dot{s}) \in \Gamma\}$ , from which it follows that  $\|(q, \dot{q}, s, \dot{s})\|_{\bar{\Gamma}} = \|(q - Ls, \dot{q} - L\dot{s})\|_{\Gamma}$ . This fact and the inequalities in (3) imply that

$$\alpha(\|(q, \dot{q}, s, \dot{s})\|_{\bar{\Gamma}}) \leq H(q - Ls, \dot{q} - L\dot{s}) \leq \beta(\|(q, \dot{q}, s, \dot{s})\|_{\bar{\Gamma}}). \quad (30)$$

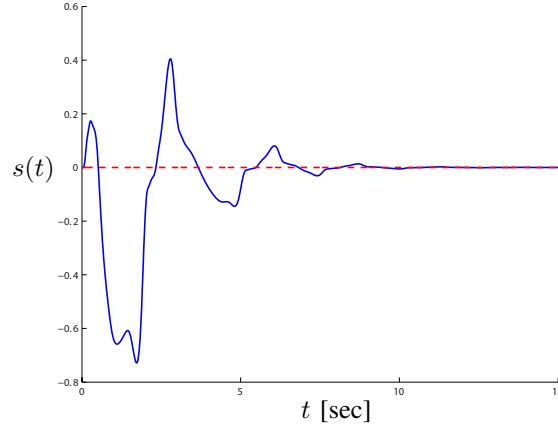


Fig. 7. Output of the double integrator.

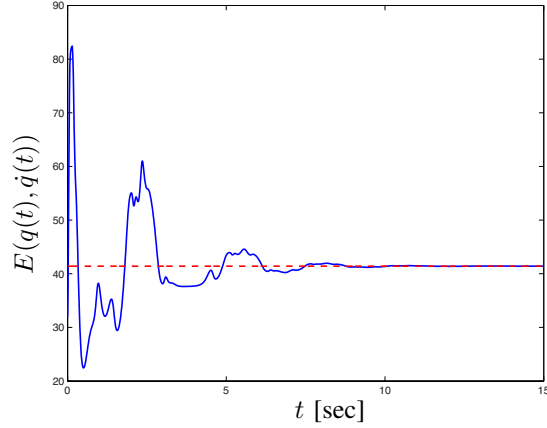


Fig. 8. Energy of the vehicle on the constraint manifold.

Letting  $e = h(q - Ls)$ , the feedback (15) gives  $\ddot{e} + k_2\dot{e} + k_1e = 0$ , so that the equilibrium  $(e, \dot{e}) = (0, 0)$  is asymptotically stable. Since  $(e, \dot{e}) = H(q - Ls, \dot{q} - L\dot{s})$ , property (30) implies that  $\bar{\Gamma}$  is asymptotically stable.  $\square$

Let  $H : \mathcal{X} \rightarrow \mathbb{R}^{n-1}$  and  $\pi : \mathcal{U} \rightarrow [\mathbb{R}]_T$  be as in the theorem statement. We claim that there exists a neighborhood  $\mathcal{V}$  of  $\gamma$  in  $\mathcal{X}$  such that the map  $F : \mathcal{V} \rightarrow [\mathbb{R}]_T \times \mathbb{R}^{n-1}$ ,  $x \mapsto (\vartheta, z) = (\pi(x), H(x))$  is a diffeomorphism onto its image. By the generalized inverse function theorem [10], we need to show that  $dF_x$  is an isomorphism for each  $x \in \gamma$ , and that  $F|_\gamma$  is a diffeomorphism  $\gamma \rightarrow [\mathbb{R}]_T \times \{0\}$ . The first property was proved in [12, Proposition 1.2]. For the second property, we observe that  $F|_\gamma = \pi|_\gamma \times \{0\}$  is a diffeomorphism  $\gamma \rightarrow [\mathbb{R}]_T \times \{0\}$ , since  $\pi|_\gamma = \varphi^{-1}$  is a diffeomorphism  $\gamma \rightarrow [\mathbb{R}]_T$ . The smooth inverse of  $F|_\gamma$  is

$$(F|_\gamma)^{-1} = F^{-1}(\vartheta, 0) = \varphi(\vartheta). \quad (31)$$

Thus  $F : \mathcal{V} \rightarrow [\mathbb{R}]_T \times \mathbb{R}^{n-1}$  is a diffeomorphism onto its image, as claimed. Since  $\vartheta \mapsto \varphi(\vartheta)$  is a regular parameterization of the orbit  $\gamma$ , and since  $\gamma$  is an invariant set for the open-loop system,  $f(\varphi(\vartheta))$  is proportional to  $\varphi'(\vartheta)$ . More precisely, defining the continuous function  $[\mathbb{R}]_T \rightarrow \mathbb{R}$ ,

$$\rho(\vartheta) = \frac{\langle f(\varphi(\vartheta)), \varphi'(\vartheta) \rangle}{\|\varphi'(\vartheta)\|^2}, \quad (32)$$

we have that

$$(\forall \vartheta \in [\mathbb{R}]_T) \quad \varphi'(\vartheta) = \frac{1}{\rho(\vartheta)} f(\varphi(\vartheta)), \quad (33)$$

and  $\rho$  is bounded away from zero. We now represent the control system (19) in  $(\vartheta, z)$  coordinates. The development is a slight variation of the one presented in the proof of [12, Proposition 1.4], the variation being due to the fact that, in [12], it is assumed that  $\rho = 1$ . For the  $\vartheta$ -dynamics, we have

$$\dot{\vartheta} = [L_f \pi(x) + L_g \pi(x)u]_{x=F^{-1}(\vartheta, z)}.$$

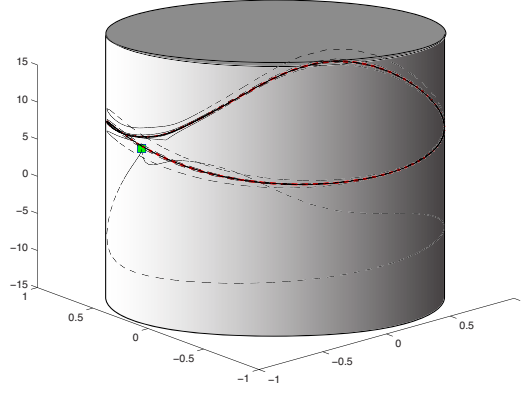


Fig. 9. The time trajectory of  $(q_2, \dot{q}_2)$  on the cylinder  $\mathbb{S}^1 \times \mathbb{R}$ .

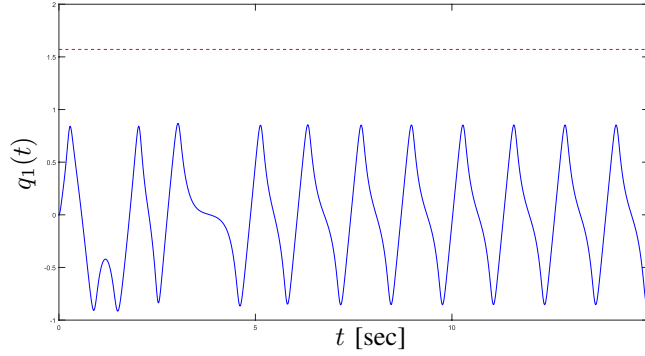


Fig. 10. The time trajectory of  $q_1$ .

We claim that the restriction of the drift term to  $\gamma$  is  $\rho(\vartheta)$ . Indeed, using (31) and (33), we have

$$[L_f \pi(x)]_{x=F^{-1}(\vartheta, 0)} = L_f \pi(\varphi(\vartheta)) = d\pi_{\varphi(\vartheta)} f(\varphi(\vartheta)) = \rho(\vartheta) d\pi_{\varphi(\vartheta)} \varphi'(\vartheta) = \rho(\vartheta).$$

The last equality is due to the fact that  $\pi(\varphi(\vartheta)) = \vartheta$ , so that  $d\pi_{\varphi(\vartheta)} \varphi'(\vartheta) = 1$ . Thus we may write

$$\dot{\vartheta} = \rho(\vartheta) + f_1(\vartheta, z) + g_1(\vartheta, z)u,$$

where  $f_1(\vartheta, 0) = 0$ . The derivation of the  $z$  dynamics is essentially the same as in [12, Proposition 1.4] so we present their form without proof. The control system (19) in  $(\vartheta, z)$  coordinates has the form

$$\begin{aligned} \dot{\vartheta} &= \rho(\vartheta) + f_1(\vartheta, z) + g_1(\vartheta, z)u \\ \dot{z} &= \bar{A}(\vartheta)z + f_2(\vartheta, z) + g_2(\vartheta, z)u, \end{aligned} \tag{34}$$

where  $f_1$  and  $f_2$  satisfy  $f_1(\vartheta, 0) = 0$ ,  $f_2(\vartheta, 0) = 0$ ,  $\partial_z f_2(\vartheta, 0) = 0$ .

Letting  $\tilde{T} = \int_0^T |1/\rho(u)| du$ , we have that  $\tilde{T} > 0$  because  $\rho$  is bounded away from zero. Consider the partial coordinate transformation  $\tau : [\mathbb{R}]_T \rightarrow [\mathbb{R}]_{\tilde{T}}$  defined as

$$\tau(\vartheta) = \left[ \int_0^{\vartheta} 1/\rho(u) du \right]_{\tilde{T}}.$$

Since  $\rho$  is bounded away from zero, the derivative  $\tau'(\vartheta)$  is also bounded away from zero, implying that  $\tau$  is a diffeomorphism. We denote by  $\vartheta(\tau)$  the inverse of  $\tau(\vartheta)$ . System (34) in  $(\tau, z)$  coordinates reads as

$$\begin{aligned} \dot{\tau} &= 1 + \tilde{f}_1(\tau, z) + \tilde{g}_1(\tau, z)u \\ \dot{z} &= \bar{A}(\vartheta(\tau))z + \tilde{f}_2(\tau, z) + g_2(\vartheta(\tau), z)u, \end{aligned} \tag{35}$$

where  $\tilde{f}_1(\tau, z) = f_1(\vartheta(\tau), z)/\rho(\vartheta(\tau))$ ,  $\tilde{g}_1(\tau, z) = g_1(\vartheta(\tau), z)/\rho(\vartheta(\tau))$ , and  $\tilde{f}_2(\tau, z) = f_2(\vartheta(\tau), z)$ .

System (35) has the same form of that in [12, Proposition 1.4] (which, however, has no control inputs). By [12, Proposition 1.5], we deduce that the orbit  $\gamma$  is exponentially stabilizable if and only if the  $\tilde{T}$ -periodic system

$$\frac{dz}{d\tau} = \bar{A}(\vartheta(\tau))z + \tilde{g}_2(\vartheta(\tau), 0)u,$$

is stabilizable. Since  $\vartheta(\tau)$  is a diffeomorphism, we may perform the time-scaling

$$\frac{dz}{d\vartheta} = \frac{1}{\rho(\vartheta)} [\bar{A}(\vartheta)z + g_2(\vartheta, 0)u]. \quad (36)$$

Thus  $\gamma$  is exponentially stabilizable if and only if the  $T$ -periodic system (36) is asymptotically stable. By comparing the system and input matrices of (36) with those of system (21), we see that to prove part (a) of Theorem 10 it suffices to show that

$$\bar{A}(\vartheta) = [(dL_f H)_{\varphi(\vartheta)}] dH_{\varphi(\vartheta)}^\dagger \quad (37)$$

$$g_2(\vartheta, 0) = L_g H(\varphi(\vartheta)). \quad (38)$$

Since  $z = H(x)$ , the coefficient of  $u$  in  $\dot{z}$  is

$$g_2(\vartheta, z) = L_g H \circ F^{-1}(\vartheta, z).$$

Using (31) we get  $g_2(\vartheta, 0) = L_g H \circ F^{-1}(\vartheta, 0) = L_g H(\varphi(\vartheta))$ . This proves identity (38).

Concerning identity (37), and referring to system (34),  $\bar{A}(\vartheta)$  is the Jacobian of  $\dot{z}$  with respect to  $z$  evaluated at  $(z, u) = (0, 0)$ . Since

$$\dot{z} = L_f H \circ F^{-1}(\vartheta, z) + L_g H \circ F^{-1}(\vartheta, z)u,$$

we have

$$\bar{A}(\vartheta) = \partial_z [L_f H \circ F^{-1}(\vartheta, z)]|_{z=0}.$$

By the chain rule and the identity (31), we get

$$\bar{A}(\vartheta) = [(dL_f H)_{\varphi(\vartheta)}] \partial_z F^{-1}(\vartheta, z)|_{z=0}.$$

To show that identity (37) holds, we need to show that  $\partial_z F^{-1}(\vartheta, z)|_{z=0} = dH_{\varphi(\vartheta)}^\dagger$ . To this end, we use the fact that

$$dF_{\varphi(\vartheta)} dF_{(\vartheta, 0)}^{-1} = I_n,$$

or

$$\begin{bmatrix} d\pi_{\varphi(\vartheta)} \\ dH_{\varphi(\vartheta)} \end{bmatrix} [\partial_\vartheta F^{-1} \quad \partial_z F^{-1}(\vartheta, z)]|_{z=0} = I_n.$$

In light of the above,  $\partial_z F^{-1}(\vartheta, z)|_{z=0}$  is uniquely defined by the identities

$$\begin{aligned} d\pi_{\varphi(\vartheta)} \partial_z F^{-1}(\vartheta, z)|_{z=0} &= 0 \\ dH_{\varphi(\vartheta)} \partial_z F^{-1}(\vartheta, z)|_{z=0} &= I_{n-1}, \end{aligned}$$

so we need to show that

$$d\pi_{\varphi(\vartheta)} dH_{\varphi(\vartheta)}^\dagger = 0 \quad (39)$$

$$dH_{\varphi(\vartheta)} dH_{\varphi(\vartheta)}^\dagger = I_{n-1}, \quad (40)$$

Identity (40) holds by virtue of the fact that  $dH^\dagger$  is the right-inverse of  $dH$ . Using the definition of pseudoinverse and taking the transpose of (39), we may rewrite (39) as

$$dH_{\varphi(\vartheta)} d\pi_{\varphi(\vartheta)}^\top = 0.$$

Since  $\varphi(\pi(x)) = x$  for all  $x \in \gamma$ , we have  $d\varphi_{\pi(x)} d\pi_x = I_n$ , or

$$(\forall x \in \gamma) \quad d\pi_x^\top = \frac{d\varphi_{\pi(x)}}{\|d\varphi_{\pi(x)}\|_2^2},$$

so that

$$dH_{\varphi(\vartheta)} d\pi_{\varphi(\vartheta)}^\top = \frac{dH_{\varphi(\vartheta)} d\varphi_\vartheta}{\|d\varphi_\vartheta\|_2^2}.$$

Since  $H(\varphi(\vartheta)) \equiv 0$ ,  $dH_{\varphi(\vartheta)} d\varphi_\vartheta = 0$  for all  $\vartheta \in [\mathbb{R}]_T$ . Thus,  $dH_{\varphi(\vartheta)} d\pi_{\varphi(\vartheta)}^\top = 0$  for all  $\vartheta \in [\mathbb{R}]_T$ .

We have thus shown that identities (39) and (40) hold, implying that identity (37) holds. This concludes the proof of part (a) of the theorem.

For part (b), let  $A(t)$ ,  $B(t)$  be as in (21), and suppose that the origin of  $\dot{z} = (A(t) + B(t)K(t))z$  is asymptotically stable. With the controller  $u^*(x) = K(\pi(x))H(x)$ , the dynamics of the closed-loop system in  $(\tau, z)$  coordinates read as

$$\begin{aligned} \dot{\tau} &= 1 + \tilde{f}_1(\tau, z) + \tilde{g}_1(\tau, z)K(\vartheta(\tau))z \\ \dot{z} &= \bar{A}(\vartheta(\tau)) + \tilde{f}_2(\tau, z) + g_2(\vartheta(\tau), z)K(\vartheta(\tau))z. \end{aligned} \quad (41)$$

For the  $z$  dynamics we have

$$\begin{aligned}\dot{z} &= [\bar{A}(\vartheta(\tau)) + g_2(\vartheta(\tau), 0)K(\vartheta(\tau))]z + \tilde{f}_2(\tau, z) + [g_2(\vartheta(\tau), z) - g_2(\vartheta(\tau), 0)]K(\vartheta(\tau))z \\ &= [\bar{A}(\vartheta(\tau)) + g_2(\vartheta(\tau), 0)K(\vartheta(\tau))]z + \tilde{F}_2(\vartheta, z),\end{aligned}\quad (42)$$

with  $\tilde{F}_2(\vartheta, 0) = 0$ ,  $\partial_z \tilde{F}_2(\vartheta, 0) = 0$ . By using  $\vartheta$  as time variable, the linear part of the  $z$ -dynamics reads as

$$\frac{dz}{d\vartheta} = \frac{1}{\rho(\vartheta)} [\bar{A}(\vartheta) + g_2(\vartheta, 0)K(\vartheta)]z.$$

Using the identities (37) and (38) we rewrite the above as

$$\frac{dz}{d\vartheta} = (A(\vartheta) + B(\vartheta)K(\vartheta))z.$$

By assumption, the origin of this system is asymptotically stable, implying that the origin of the system  $\dot{z} = [\bar{A}(\vartheta(\tau)) + g_2(\vartheta(\tau), 0)K(\vartheta(\tau))]z$  has the same property. Referring to (42) and using [12, Proposition 1.5], we conclude that the closed orbit  $\gamma$  is exponentially stable for the closed-loop system (41) and hence also for system (19) with feedback  $u^*(x) = K(\pi(x))H(x)$ .  $\square$

**Acknowledgments** A. Mohammadi and M. Maggiore were supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada. A. Mohammadi was partially supported by the University of Toronto Doctoral Completion Award (DCA).

## REFERENCES

- [1] M. Ahmed, A. Hably, and S. Bacha. Kite generator system periodic motion planning via virtual constraints. In *IEEE Conf. Ind. Elect.*, pages 1–6, 2013.
- [2] S. Bittanti, P. Colaneri, and G. De Nicolao. The periodic Riccati equation. In *The Riccati Equation*, pages 127–162. Springer, 1991.
- [3] C. Canudas-de Wit. On the concept of virtual constraints as a tool for walking robot control and balancing. *Annual Reviews in Control*, 28(2):157–166, 2004.
- [4] C. Canudas-de Wit, B. Espiau, and C. Urrea. Orbital stabilization of underactuated mechanical systems. In *Proceedings of the 15th IFAC World Congress. Barcelona*, 2002.
- [5] L. Consolini and M. Maggiore. On the swing-up of the pendubot using virtual holonomic constraints. In *IFAC World Congress*, Milano, Italy, 2011.
- [6] L. Consolini, M. Maggiore, C. Nielsen, and M. Tosques. Path following for the PVTOL aircraft. *Automatica*, 46(8):1284–1296, 2010.
- [7] M. El-Hawwary and M. Maggiore. Reduction theorems for stability of closed sets with application to backstepping control design. *Automatica*, 49(1):214–222, 2013.
- [8] J. W. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Trans. Automat. Contr.*, 46(1):51–64, 2001.
- [9] J. W. Grizzle, C. Chevallereau, R. W. Sinnet, and A. D. Ames. Models, feedback control, and open problems of 3d bipedal robotic walking. *Automatica*, 50(8):1955–1988, 2014.
- [10] V. Guillemin and A. Pollack. *Differential Topology*. Prentice Hall, New Jersey, 1974.
- [11] J. K. Hale. *Ordinary Differential Equations*. Robert E. Krieger Publishing Company, second edition, 1980.
- [12] J. Hauser and C.C. Chung. Converse lyapunov functions for exponentially stable periodic orbits. *Systems & Control Letters*, 23(1):27–34, 1994.
- [13] J. Hauser, S. Sastry, and G. Meyer. Nonlinear control design for slightly non-minimum phase systems: Applications to V/STOL aircraft. *Automatica*, 28(4):665–679, 1992.
- [14] J. M. Lee. *Introduction to Smooth Manifolds*. Springer, second edition, 2013.
- [15] M. Maggiore and L. Consolini. Virtual holonomic constraints for Euler-Lagrange systems. *IEEE Trans. Automat. Contr.*, 58(4):1001–1008, 2013.
- [16] A. Mohammadi, M. Maggiore, and L. Consolini. When is a Lagrangian control system with virtual holonomic constraints Lagrangian? In *Proc. the 9th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, Toulouse, France, 2013.
- [17] A. Mohammadi, M. Maggiore, and L. Consolini. On the Lagrangian structure of reduced dynamics under virtual holonomic constraints. *ESAIM: Control, Optimisation, and Calculus of Variations*, 2016. doi: 10.1051/cocv/2016020.
- [18] A. Mohammadi, E. Rezapour, M. Maggiore, and K. Y. Pettersen. Direction following control of planar snake robots using virtual holonomic constraints. In *53rd Conf. Decision and Control (CDC)*, pages 3801–3808, 2014.
- [19] A. Mohammadi, E. Rezapour, M. Maggiore, and K. Y. Pettersen. Maneuvering control of planar snake robots using virtual holonomic constraints. *IEEE Trans. Contr. Syst. Technol.*, 24(3):884–899, 2016.
- [20] B. Morris and J. W. Grizzle. Hybrid invariant manifolds in systems with impulse effects with application to periodic locomotion in bipedal robots. *IEEE Trans. Automat. Contr.*, 54(8):1751–1764, 2009.
- [21] E. Rezapour, A. Hofmann, K. Y. Pettersen, A. Mohammadi, and M. Maggiore. Virtual holonomic constraint based direction following control of planar snake robots described by a simplified model. In *IEEE Conf. Control Applications*, pages 1064–1071, 2014.
- [22] P. Seibert and J. S. Florio. On the reduction to a subspace of stability properties of systems in metric spaces. *Annali di Matematica pura ed applicata*, 169(1):291–320, 1995.
- [23] A. S. Shiriaev, L. B. Freidovich, A. Robertsson, R. Johansson, and A. Sandberg. Virtual-holonomic-constraints-based design of stable oscillations of furuta pendulum: Theory and experiments. *IEEE Trans. Robot.*, 23(4):827–832, 2007.
- [24] A. S. Shiriaev, J. W. Perram, and C. Canudas-de-Wit. Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Trans. Automat. Contr.*, 50(8):1164–1176, August 2005.
- [25] A.S. Shiriaev, L.B. Freidovich, and S.V. Gusev. Transverse linearization for controlled mechanical systems with several passive degrees of freedom. *IEEE Trans. Automat. Contr.*, 55(4):893–906, 2010.
- [26] E.R. Westervelt, J.W. Grizzle, C. Chevallereau, J.H. Choi, and B. Morris. *Feedback Control of Dynamic Bipedal Robot Locomotion*. Taylor & Francis, CRC Press, 2007.